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Recursion relations and branching rules
for simple Lie algebras

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Abstract.

The branching rules between simple Lie algebras and its regular (maximal) simple subalgebras are studied. Two types of recursion relations for anomalous relative multiplicities are obtained. One of them is proved to be the factorized version of the other. The factorization property is based on the existence of the set of weights Γ specific for each injection. The structure of Γ is easily deduced from the correspondence between the root systems of algebra and subalgebra. The recursion relations thus obtained give rise to simple and effective algorithm for branching rules. The details are exposed by performing the explicit decomposition procedure for $A_3 \oplus u(1) \to B_4$ injection.

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1   Introduction

1.1

In elementary particle physics and especially in model building it is quite important to have effective branching rules for Lie algebras representations. There are several simple methods of decomposition appropriate for different types of injections, for example, the Gelfand-Zetlin method for $A_{n-1} \to A_n, B_{n-1} \to B_n$ and $D_{n-1} \to D_n$ [1, 2] . In the general case the most advanced investigation was performed by R.V.Moody, J.Patera,R.T.Sharp and F.Gingras in a series of works [3, 4, 5]. Their approach is based on the properties of Weyl orbits in weight diagrams and the generating function technique [6].

In this paper we want to demonstrate that recursion relations for multiplicities of subrepresentations can also be successfully used in the decomposition procedure. We restrict the exposition to regular maximal injections of reductive subalgebras (composed of semisimple ones and diagonalizable Abelian algebras), though it is possible to treat analagously special injections and also nonmaximal ones. For a simple algebra g and its regular maximal reductive subalgebra $\widetilde{g}$ the problem is to evaluate the coefficients n_μ in the decomposition of an irreducible representation $L^\lambda(g)$ (λ is its highest weight)

$$L^\lambda(g)_{\downarrow \widetilde{g}} = \oplus_\mu n_\mu \widetilde{L}^\mu(\widetilde{g}). \tag{1}$$

For any weight ν of L^λ the total multiplicity m_ν can be presented as the sum

$$m_\nu = m'_\nu + n_\nu \tag{2}$$

where m'_ν is the multiplicity induced by the subrepresentation $\oplus_{\mu > \nu} n_\mu \widetilde{L}^\mu(\widetilde{g})$ contained in (1). The second term n_ν in (2) is called *the relative multiplicity* of the weight ν. When ν is from the dominant Weyl chamber its relative multiplicity coinsides with the corresponding coefficient in the decomposition (1). The recursion relations for relative multiplicities for the injection $A_3 \oplus u(1) \to D_4$ were studied in [7]. It was shown that the considerable set of multiplicities for intermediate weights mutually cancel and the final recursion formula is suitable for calculations. As will be demonstrated below the general recursion relation for regular injections can be naturally formulated in terms of *anomalous relative multiplicities* $\widetilde{n_\nu}$. To define them consider the highest weights

$$M = \{\mu \mid n_\mu \neq 0\}$$

for the decomposition (1). For the subalgebra $\widetilde{g}$ let V be the Weyl group and $\widetilde{\rho}$ – the half-sum of positive roots. The anomalous relative multiplicity $\widetilde{n_\nu}(g, \widetilde{g}, \lambda, \nu)$ is the function

$$\widetilde{n}(g, \widetilde{g}, \lambda, \nu) \equiv \widetilde{n_\nu} = \begin{cases} det(v) n_\mu & \text{for } \{\mu \in M \mid v(\mu + \widetilde{\rho}) - \widetilde{\rho} = \nu\} \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

defined on the weight space of g. As will be demonstrated the general recursion relation for regular injections can be naturally formulated in terms of anomalous relative multiplicities $\widetilde{n_\nu}$.

The paper is organized as follows. The general formalism is presented in section 2. The obtained recursion relations are based on the properties of the elementary "fan" Γ – the special set of weights defined by the injection $\widetilde{g} \rightarrow g$. The structure of Γ's and their basic properties are studied in details. To demonstrate explicitly the role of Γ in recursion procedure we use the very simple example – the injection $A_1 \oplus u(1) \rightarrow B_2$. In the appendix the applications of the recursion formulas are shown in full details for the injection $A_3 \oplus u(1) \rightarrow B_4$.

1.2 Notations

g – the simple Lie algebra;

$\widetilde{g}$ – the reductive regular subalgebra of g;

Δ, $\widetilde{\Delta}$ – the corresponding sets of positive roots, note that $\widetilde{\Delta}$ is the system of positive roots of the semisimple subalgebra in $\widetilde{g}$;

S, $\widetilde{S}$ – the sets of basic roots;

ρ, $\widetilde{\rho}$ – the half-sums of positive roots for g and $\widetilde{g}$ respectively;

W, V – the Weyl groups for Δ and $\widetilde{\Delta}$;

$\epsilon(w)$, $\epsilon(v)$ – the determinants of the Weyl reflections w and v;

C, $\widetilde{C}$ – the Weyl chambers dominant with respect to S and $\widetilde{S}$;

$\overline{C}$, $\overline{\widetilde{C}}$ – the closures of the corresponding Weyl chambers;

P_g, $P_{\widetilde{g}}$ – the weight lattices for g and $\widetilde{g}$;

$\mathcal{E}$, $\widetilde{\mathcal{E}}$ – the formal algebras associated to the weight lattices, $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are generated by the elements e^β, where β is the fundamental weight, and the composition $e^\beta \cdot e^\gamma = e^{\beta+\gamma}$;

$ch\, L^\lambda$ – the formal character of the representation L^λ;

Ψ^λ, $\widetilde{\Psi^\mu}$ – the elements of formal algebras associated to the sets of anomalous weights for representations L^λ, $\widetilde{L^\mu}$:

$$\Psi^\lambda = \sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)-\rho}, \tag{4}$$

$$\widetilde{\Psi^\mu} = \sum_{v \in V} \epsilon(v) e^{v(\mu+\widetilde{\rho})-\widetilde{\rho}}. \tag{5}$$

For roots and weights of simple Lie algebras we use the standard e-basis [9].

2 Recursion Relations for Regular Injections

2.1

The initial decomposition (1) can be rewritten in terms of formal characters [11]

$$ch\ L^\lambda = \sum_\mu n_\mu ch\ L^\mu. \tag{6}$$

Applying the Weyl formula [10]

$$ch\ L^\xi = \frac{\Psi^\xi}{\prod_{\alpha\in\Delta}(1-e^{-\alpha})} \tag{7}$$

and taking into account the injection $\widetilde{\Delta} \to \Delta$ one gets the relation between the anomalous elements Ψ^λ and $\widetilde{\Psi^\mu}$

$$(\Pi_{\Delta\setminus\widetilde{\Delta}}(1-e^{-\alpha}))^{-1}\Psi^\lambda = \sum_\mu n_\mu \widetilde{\Psi^\mu}. \tag{8}$$

Using the basis $\{e^\xi\}_{\xi\in P_g}$ of algebra $\mathcal{E}$ one can expand both sides of the relation (8). In the left-hand side of the expansion of the first factor

$$(\Pi_{\Delta\setminus\widetilde{\Delta}}(1-e^{-\alpha}))^{-1} = \sum_\xi K_{\widetilde{g}\subset g}(\xi)e^{-\xi} \tag{9}$$

gives rise to the so-called Kostant-Heckman partition function [8]. The expression (9) together with the formulas (4) and (5) gives the desired expansion of (8). Now consider only the weights μ from the dominant Weyl chamber $\widetilde{C}$, that is the anomalous weights with $v=e$. Comparing the coefficients one gets the expression for the relative multiplicity n_μ in terms of partition function $K_{\widetilde{g}\subset g}$:

$$n_\mu = \sum_W \epsilon(w)K_{\widetilde{g}\subset g}(w(\lambda+\rho)-(\rho+\mu)). \tag{10}$$

Note that the relation (8) is valid on the whole weight lattice P_g. Thus one can rewrite it in the form:

$$\sum_\mu n_\mu\widetilde{\Psi^\mu} = \sum_\mu n_\mu \sum_{v\in V}\epsilon(v)e^{v(\mu+\widetilde{\rho})-\widetilde{\rho}} = \sum_\xi \widetilde{n_\xi}e^\xi. \tag{11}$$

Since all the weights $\{v(\mu+\widetilde{\rho})-\widetilde{\rho}\}$ are different the coefficients $\widetilde{n_\xi}$ here are just the anomalous relative multiplicities (see (3)). The relation (11) together with (9) and (4) shows that the expression (10) is true in all points of P_g when n is changed by $\widetilde{n}$:

$$\widetilde{n_\xi} = \sum_W \epsilon(w)K_{\widetilde{g}\subset g}(w(\lambda+\rho)-(\rho+\xi)). \tag{12}$$

We would use this formula to construct the recursion relation for $\widetilde{n_\xi}$. First consider the expression (12) for $\lambda = 0$. In this case $\widetilde{n_\xi}$ can be written explicitly as the multiplicity of anomalous weight diagram for the trivial subrepresentation $\widetilde{L^0}$:

$$\widetilde{n_\xi} = \sum_V \epsilon(v)\delta_{\nu,\,v\widetilde{\rho}-\widetilde{\rho}} = \sum_W \epsilon(w)K_{\widetilde{g}\subset g}(w\rho-(\rho+\xi)). \tag{13}$$

One can extract the trivial term (with $w = e$) and obtain the recursion relation for the partition function $K_{\widetilde{g} \subset g}$

$$K_{\widetilde{g} \subset g}(\xi) = -\sum_{W \backslash e} K_{\widetilde{g} \subset g}(\xi + (w-1)\rho) + \sum_V \epsilon(v)\delta_{\xi, \rho - v\rho}. \tag{14}$$

Returning to the expression (12) one easily comes to the conclusion that the relation (14) induces the recursion relation for anomalous relative multiplicities

$$\widetilde{n_\xi} = -\sum_{W \backslash e} \epsilon(w)\widetilde{n}_{\xi + (1-w)\rho} + \sum_{W,V} \epsilon(w)\epsilon(v)\delta_{\xi + (1-v)\widetilde{\rho}, \, w(\lambda+\rho) - \rho}. \tag{15}$$

This relation can be used in the explicit calculations of multiplicities $\widetilde{n}_\nu$ and n_μ and in some cases it is effective. But the necessity to estimate at each step the full set of probe weights $\{\xi + (1-w)\rho\}$ can make the whole process quite cumbersome. Consider once more the formula (8). The previous derivation was based on the properties of the operator $(\prod_{\Delta \backslash \widetilde{\Delta}} (1 - e^{-\alpha}))^{-1}$. Now we shall use its inverse and taking into account (11) rewrite the relation (8) in the following form

$$\Psi^\lambda = \sum_{w \in W} \epsilon(w)e^{w(\lambda+\rho)-\rho} = \prod_{\Delta \backslash \widetilde{\Delta}}(1 - e^{-\alpha}) \cdot \sum_\xi \widetilde{n_\xi} e^\xi. \tag{16}$$

The first factor in (16) defines the finite set of weights $\Gamma(g \subset \widetilde{g})$ whose structure depends only on the injection $\widetilde{g} \subset g$,

$$\prod_{\Delta \backslash \widetilde{\Delta}}(1 - e^{-\alpha}) = 1 - \sum_{\gamma \in \Gamma} \text{sign}(\gamma)e^{-\gamma}. \tag{17}$$

In these terms the formula (16) leads to the following recursion relation

$$\widetilde{n}_\nu = \sum_{\gamma \in \Gamma} \text{sign}(\gamma)\widetilde{n}_{\nu + \gamma} + \sum_{w \in W} \epsilon(w)\delta_{\nu, \, w(\lambda+\rho)-\rho}. \tag{18}$$

Its efficiency mainly depends on the possibility to construct explicitly the set $\Gamma(g \subset \widetilde{g})$.

2.2

To reveal the structure of Γ let us use the denominator identity [10],

$$\prod_{\alpha \in \Delta}(1 - e^{-\alpha}) = \Psi^0 \tag{19}$$

to transform the expression (17):

$$\prod_{\Delta \backslash \widetilde{\Delta}}(1 - e^{-\alpha}) = 1 - \sum_{\gamma \in \Gamma} \text{sign}(\gamma)e^{-\gamma} = (\prod_{\alpha \in \widetilde{\Delta}}(1 - e^{-\alpha}))^{-1} \cdot \Psi^0 \tag{20}$$

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The anomalous element Ψ^0 is W-invariant and can be factorized with respect to $V \subset W$,

$$\Psi^0 = \sum_{x \in X} \sum_{v \in V} \epsilon(v \cdot x) e^{(v \cdot x - 1)\rho}. \tag{21}$$

Here X is the factorspace W/V. This allows to present Γ as the set of weight diagrams of representations $\widetilde{L}$:

$$1 - \sum_{\gamma \in \Gamma} \text{sign}(\gamma) e^{-\gamma} = \sum_{x \in X} \epsilon(x) \Big(\prod_{\alpha \in \widetilde{\Delta}} (1 - e^{-\alpha}) \Big)^{-1} \sum_{v \in V} \epsilon(v) e^{(vx\rho - \widetilde{\rho} + (\widetilde{\rho} - \rho))} = e^{\widetilde{\rho} - \rho} \sum_{x \in X} \epsilon(x) ch\, \widetilde{L}^{(x\rho - \widetilde{\rho})} \tag{22}$$

The element Ψ^0 multiplied by $(\prod_{\alpha \in \Delta}(1 - e^{-\alpha}))^{-1}$ gives the weight of the trivial representation L^0 of g, while the same Ψ^0 multiplied by $(\prod_{\alpha \in \widetilde{\Delta}}(1 - e^{-\alpha}))^{-1}$ generates the assembly $\Xi(\widetilde{g} \subset g)$ of weight diagrams for representations of $\widetilde{g}$. To construct $\Xi(\widetilde{g} \subset g)$ one can use the auxiliary set $\Omega(\widetilde{g} \subset g)$

$$\Omega = \{0, \alpha_{i_1}, \alpha_{i_1} + \alpha_{i_2}, \ldots, \alpha_{i_1} + \ldots + \alpha_{i_m} \mid \alpha_{i_l} \in \Delta \setminus \widetilde{\Delta}, m = \text{card}(\Delta \setminus \widetilde{\Delta})\}.$$

Fix the subset Ω' of dominant weights

$$\Omega' = \{\omega \in \Omega \mid \omega \in \overline{\overline{C}}\}.$$

Equip every $\omega \in \Omega$ with the sign $\delta(\omega) = \delta(\alpha_{i_1} + \ldots + \alpha_{i_k}) = (-1)^{k+1}$. Reduce Ω' to Ω'_r cancelling every pair of weights in Ω' that has opposite signs. The subset of maximal weights in Ω'_r is just the desired assembly of representations $\Xi(\widetilde{g} \subset g)$ presented by their highest weights. Obviously each $\xi \in \Xi(\widetilde{g} \subset g)$ has the form $\xi = x\rho - \rho$ and it is easily seen that $\delta(\xi) = \epsilon(x)$.

Let Φ^ξ be the weight diagram of the irrep $\widetilde{L}^\xi$ with the highest weight $\xi \in \Xi(\widetilde{g} \subset g)$. Due to the relation (22) the set $\Gamma(\widetilde{g} \subset g)$ is obtained as the union of diagrams Φ^ξ

$$\Gamma = \Big(\bigcup_{\xi \in \Xi(\widetilde{g} \subset g)} \Phi^\xi \Big) \setminus \{0\}. \tag{23}$$

Thus to find $\Gamma(\widetilde{g} \subset g)$ it is sufficient to construct $\Xi(\widetilde{g} \subset g)$. As we have seen the latter depends only on the structure of the factorspace W/V and the weights ρ and $\widetilde{\rho}$.

Now we shall illustrate the situation treating different types of regular maximal injections. For the injection $A_{n-1} \oplus u(1) \to A_n$ the dimension of W/V gives $\text{card}(\Xi) = (n+1)!/n! = n+1$. In the cases $A_{n-1} \oplus u(1) \to B_n, C_n, D_n$ the number of weights in Ξ is proportional to powers of "2":

$$\text{card}(\Xi) = \begin{cases} 2^n & \text{for } A_{n-1} \oplus u(1) \to B_n, \\ 2^n & \text{for } A_{n-1} \oplus u(1) \to C_n, \\ 2^{n-1} & \text{for } A_{n-1} \oplus u(1) \to D_n. \end{cases}$$

The elementary analysis of the orbits of the Weyl group V on the space of faithful representation of W generated by the weight ρ leads to the following results.

Lemma 1 *Put $\alpha_0 \equiv (0, \ldots, 0)$ and let $\{\alpha_1, \ldots, \alpha_s\}$ be the ordered sequences of roots:*

$$\{e_1 - e_{n+1}, e_2 - e_{n+1}, \ldots, e_n - e_{n+1}\} \quad for \;\; \Delta_{A_n} \setminus \Delta_{A_{n-1}},$$

$$\{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n, e_1,$$
$$e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\} \qquad for \;\; \Delta_{B_n} \setminus \Delta_{B_{n-1}},$$

$$\{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n, 2e_1,$$
$$e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\} \qquad for \;\; \Delta_{C_n} \setminus \Delta_{C_{n-1}},$$

$$\{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n,$$
$$e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n\} \qquad for \;\; \Delta_{D_n} \setminus \Delta_{D_{n-1}};$$

then the set Ξ for $A_{n-1} \oplus u(1) \to A_n$ contains the weights

$$\xi_k = \sum_{j=0}^{k} \alpha_j; \quad k = 0, \ldots, n$$

while for $B_{n-1} \oplus u(1) \to B_n$ and $C_{n-1} \oplus u(1) \to C_n$

$$\xi_k = \sum_{j=0}^{k} \alpha_j; \quad k = 0, \ldots, 2n - 1$$

and for $D_{n-1} \oplus u(1) \to D_n$

$$\xi_k = \sum_{j=0}^{k} \alpha_j; \quad k = 0, \ldots, 2n - 2$$

$$\xi_{2n-1} = (n - 1, 1, \ldots, 1, -1).$$

Lemma 2 *In the sets $\Xi(A_{n-1} \oplus u(1) \to B_n, C_n, D_n)$ the zeroth weight is trivial*

$$\xi_0 = (0, \ldots, 0);$$

the first weight has the form

$$\xi_1 = (p_1, p_2, \ldots, p_n) = \begin{cases} (1, 0, 0, \ldots, 0) & for \;\; A_{n-1} \oplus u(1) \to B_n, \\ (2, 0, 0, \ldots, 0) & for \;\; A_{n-1} \oplus u(1) \to C_n, \\ (1, 1, 0, \ldots, 0) & for \;\; A_{n-1} \oplus u(1) \to D_n; \end{cases}$$

while the others can be obtained from the following relations:

$$\xi_{2^k} = (\underbrace{p_1 + k, 1, 1, \ldots, 1}_{k+l}, 0, \ldots, 0), \quad l = \begin{cases} 1 \begin{cases} for \;\; A_{n-1} \oplus u(1) \to B_n \\ for \;\; A_{n-1} \oplus u(1) \to C_n \end{cases} \\ 2 \;\; for \;\; A_{n-1} \oplus u(1) \to D_n \end{cases},$$

$$\xi_m = \xi_{2^k + i} = \xi_{2^k} + (\xi_i)_{shift};$$

$$i = 1, \ldots, 2^k - 1;$$

where for every $\xi_i = (q_1, q_2, \ldots, q_n)$ the shifted weight $(\xi_i)_{shift}$ is defined with the coordinates

$$(\xi_i)_{shift} = (0, q_1, q_2, \ldots, q_{n-1}).$$

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Note that in the framework of rules described in Lemma 1 and Lemma 2 the sets Ξ are totally defined by the first weight ξ_1.

Now let us accumulate the information about Ξ's for the described types of injections in the following tables.

$\widetilde{g} \to g$	The set $\Xi(\widetilde{g} \subset g)$ in terms of		sign $\gamma(\xi)$
	highest weights ξ	Dynkin indices of $\widetilde{L^\xi}$	
$A_{n-1} \oplus u(1)$ $\to A_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 0, \ldots, 0, -1)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(1, 1, 0, \ldots, 0, -2)$	$([0, 1, \ldots, 0], 2)$	$-$
	$\ldots$	$\ldots$	$\ldots$
	$(1, 1, \ldots, 1, -n)$	$([0, 0, \ldots, 0], n)$	$(-1)^{n+1}$
$A_{n-1} \oplus u(1)$ $\to B_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 0, \ldots, 0)$	$([1, 0, \ldots, 0], 1)$	$+$
	$(2, 1, 0, \ldots, 0)$	$([1, 1, 0, \ldots, 0], 3)$	$-$
	$(2, 2, 0, \ldots, 0)$	$([0, 2, 0, \ldots, 0], 4)$	$+$
	$(3, 1, 1, 0, \ldots, 0)$	$([2, 0, 1, 0, \ldots, 0], 5)$	$+$
	$(3, 2, 1, 0, \ldots, 0)$	$([1, 1, 1, 0, \ldots, 0], 6)$	$-$
	$(3, 3, 2, 0, \ldots, 0)$	$([0, 1, 2, 0, \ldots, 0], 8)$	$+$
	$(3, 3, 3, 0, \ldots, 0)$	$([0, 0, 3, 0, \ldots, 0], 9)$	$-$
	$\ldots$	$\ldots$	$\ldots$
	$(n, n, \ldots, n, n-1)$	$([0, 0, \ldots, 0, 1], n^2 - 1)$	$(-1)^{1/2(n^2+n)}$
	$(n, n, \ldots, n)$	$([0, 0, \ldots, 0], n^2)$	$(-1)^{1/2(n^2+n+2)}$
$A_{n-1} \oplus u(1)$ $\to C_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(2, 0, \ldots, 0)$	$([2, 0, \ldots, 0], 2)$	$+$
	$(3, 1, 0, \ldots, 0)$	$([2, 1, 0, \ldots, 0], 4)$	$-$
	$(3, 3, 0, \ldots, 0)$	$([0, 3, 0, \ldots, 0], 6)$	$+$
	$(4, 1, 1, 0, \ldots, 0)$	$([3, 0, 1, 0, \ldots, 0], 6)$	$+$
	$(4, 3, 1, 0, \ldots, 0)$	$([1, 2, 1, 0, \ldots, 0], 8)$	$-$
	$(4, 4, 2, 0, \ldots, 0)$	$([0, 2, 2, 0, \ldots, 0], 10)$	$+$
	$(4, 4, 4, 0, \ldots, 0)$	$([0, 0, 4, 0, \ldots, 0], 12)$	$-$
	$\ldots$	$\ldots$	$\ldots$
	$(n+1, \ldots, n+1, n-1)$	$([0, 0, \ldots, 2], n^2 + n - 2)$	$(-1)^{1/2(n^2+n)}$
	$(n+1, \ldots, n+1)$	$([0, 0, \ldots, 0], n^2 + n)$	$(-1)^{1/2(n^2+n+2)}$
$A_{n-1} \oplus u(1)$ $\to D_n$	$(0, 0, \ldots, 0)$	$([0, 0, \ldots, 0], 0)$	
	$(1, 1, 0, \ldots, 0)$	$([0, 1, 0, \ldots, 0], 2)$	$+$
	$(2, 1, 1, 0, \ldots, 0)$	$([1, 0, 1, 0, \ldots, 0], 4)$	$-$
	$(2, 2, 2, 0, \ldots, 0)$	$([0, 0, 2, 0, \ldots, 0], 6)$	$+$
	$(3, 1, 1, 1, 0, \ldots, 0)$	$([2, 0, 0, 1, 0, \ldots, 0], 6)$	$+$
	$(3, 2, 2, 1, 0, \ldots, 0)$	$([1, 0, 1, 1, 0, \ldots, 0], 8)$	$-$
	$(3, 3, 2, 2, 0, \ldots, 0)$	$([0, 1, 0, 2, 0, \ldots, 0], 10)$	$+$
	$(3, 3, 3, 3, 0, \ldots, 0)$	$([0, 0, 0, 3, 0, \ldots, 0], 12)$	$-$
	$\ldots$	$\ldots$	$\ldots$
	$(n-1, \ldots, n-1, n-2, n-2)$	$([0, \ldots, 0, 1, 0], n^2 - n - 2)$	$(-1)^{1/2(n^2-n)}$
	$(n-1, \ldots, n-1)$	$([0, \ldots, 0], n^2 - n)$	$(-1)^{1/2(n^2-n+2)}$

Table 1

$\widetilde{g} \to g$	The set $\Xi(\widetilde{g} \subset g)$ in terms of		sign $\gamma(\xi)$
	highest weights ξ	Dynkin indices of $\widetilde{L^{\xi}}$	
	$(0,0,\ldots,0)$	$([0,0,\ldots,0],0)$	
	$(1,1,0,\ldots,0)$	$([1,0,\ldots,0],1)$	$+$
	$(2,1,1,0,\ldots,0)$	$([0,1,0,\ldots,0],2)$	$-$
	$(3,1,1,1,0,\ldots,0)$	$([0,0,1,0,\ldots,0],3)$	$+$
	$\ldots$	$\ldots$	$\ldots$
$B_{n-1} \oplus u(1)$	$(n-2,1,1,\ldots,1,0)$	$([0,0,\ldots,1,0],n-2)$	$(-1)^{n-1}$
$\to B_n$	$(n-1,1,1,\ldots,1)$	$([0,\ldots,0,2],n-1)$	$(-1)^{n}$
	$(n,1,1,\ldots,1)$	$([0,\ldots,0,2],n)$	$(-1)^{n+1}$
	$(n+1,1,1,\ldots,1,0)$	$([0,\ldots,1,0],n+1)$	$(-1)^{n+2}$
	$\ldots$	$\ldots$	$\ldots$
	$(2n-2,1,0,\ldots,0)$	$([1,0,\ldots,0],2n-2)$	$-$
	$(2n-1,0,\ldots,0)$	$([0,0,\ldots,0],2n-1)$	$+$
	$(0,0,\ldots,0)$	$([0,0,\ldots,0],0)$	
	$(1,1,0,\ldots,0)$	$([1,0,\ldots,0],1)$	$+$
	$(2,1,1,0,\ldots,0)$	$([0,1,0,\ldots,0],2)$	$-$
	$(3,1,1,1,0,\ldots,0)$	$([0,0,1,0,\ldots,0],3)$	$+$
$C_{n-1} \oplus u(1)$	$\ldots$	$\ldots$	$\ldots$
$\to C_n$	$(n-1,1,\ldots,1)$	$([0,\ldots,0,1],n-1)$	$(-1)^{n}$
	$(n+1,1,\ldots,1)$	$([0,\ldots,0,1],n+1)$	$(-1)^{n+1}$
	$(n+2,1,\ldots,1,0)$	$([0,\ldots,0,1,0],n+2)$	$(-1)^{n+2}$
	$\ldots$	$\ldots$	$\ldots$
	$(2n-1,1,0,\ldots,0)$	$([1,0,\ldots,0],2n-1)$	$-$
	$(2n,0,\ldots,0)$	$([0,\ldots,0],2n)$	$+$
	$(0,0,\ldots,0)$	$([0,0,\ldots,0],0)$	
	$(1,1,0,\ldots,0)$	$([1,0,\ldots,0],1)$	$+$
	$(2,1,1,0,\ldots,0)$	$([0,1,0,\ldots,0],2)$	$-$
	$\ldots$	$\ldots$	$\ldots$
$D_{n-1} \oplus u(1)$	$(n-3,1,\ldots,1,0,0)$	$([0,\ldots,0,1,0,0],n-3)$	$(-1)^{n-2}$
$\to D_n$	$(n-2,1,\ldots,1,0)$	$([0,\ldots,0,1,1],n-2)$	$(-1)^{n-1}$
	$(n-1,1,\ldots,1,-1)$	$([0,\ldots,0,2,0],n-1)$	$(-1)^{n}$
	$(n-1,1,\ldots,1,1)$	$([0,\ldots,0,2],n-1)$	$(-1)^{n}$
	$(n,1,\ldots,1,0)$	$([0,\ldots,0,1,1],n)$	$(-1)^{n+1}$
	$(n+1,1,\ldots,1,0,0)$	$([0,\ldots,1,0,0],n+1)$	$(-1)^{n+2}$
	$\ldots$	$\ldots$	$\ldots$
	$(2n-3,1,0,\ldots,0)$	$([1,0,\ldots,0],2n-3)$	$+$
	$(2n-2,0,\ldots,0)$	$([0,\ldots,0],2n-2)$	$-$

Table 2

To conclude the general exposition of the recursion properties of $\widetilde{n}_\mu$ we show the inter-dependence of the two recursion formulas (15) and (18).

Lemma 3 *The recursion relation (15) can be factorized with respect to the subgroup V of W so that the summation over the factorspace $W \setminus V$ is replaced by the summation over Γ.*

Proof. Use the formula (18) to write down the recursion relation for expression $\sum_{v \in V} \epsilon(w) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}}$ as a whole and extract the first term corresponding to $v = e$:

$$
\begin{aligned}
\widetilde{n}_\nu =\ & -\sum_{v \in V, v \neq e} \epsilon(w) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}} - \sum_{v \in V} \sum_{\gamma \in \Gamma} \epsilon(v)\, \mathrm{sign}(\gamma) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}+\gamma} \\
& + \sum_{w \in W, v \in V} \epsilon(w)\epsilon(v)\delta_{\nu+(1-v)\widetilde{\rho},\, w(\lambda+\rho)-\rho} \\
=\ & \sum_{V, \Gamma \cup \{0\}} \epsilon(v)\, \mathrm{sign}(\gamma) \widetilde{n}_{\nu+(1-v)\widetilde{\rho}+\gamma} + \\
& + \sum_{w \in W, v \in V} \epsilon(w)\epsilon(v)\delta_{\nu+(1-v)\widetilde{\rho},\, w(\lambda+\rho)-\rho}.
\end{aligned}
\tag{24}
$$

Comparing this expression with (15) we see that introducing Γ one provides the factorization in the first term of the relation (15). Thus the relation (18) can be called *the factorized recursion formula* for anomalous relative multiplicities.

2.3

Both formulas (15) and (18) provide the effective tools to treat the branching rules decompositions for maximal regular injections. One can easily estimate the relative capacities of these relations for different pairs of g and $\widetilde{g}$. The result is that there are five families of injections mostly favourable for the relation (1): $A_{n-1} \oplus u(1) \to A_n$, $B_{n-1} \oplus u(1) \to B_n$, $C_{n-1} \oplus u(1) \to C_n$, $D_{n-1} \oplus u(1) \to D_n$, $A_{n-1} \oplus u(1) \to B_n$. For these five types the efficiency of the formula (18) increases with growth of n in comparison with that of (15). For the first four types the ordinary decomposition methods are suitable (Gelfand- Zeitlin procedure [1], for example). So we shall concentrate our attention on the last family: $A_{n-1} \oplus u(1) \to B_n$.

To show the application of the factorized formula (18) in details we shall start with the quite simple example. Consider the injection $A_1 \oplus u(1) \to B_2$. Fix the basic roots of B_2

$$
S(B_2) = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2\}
$$

and the fundamental weights

$$
\{\beta_1 = e_1, \beta_2 = \frac{1}{2}(e_1 + e_2)\}.
$$

According to table 1 we have four highest weights in the set $\Xi(A_1 \oplus u(1) \to B_2)$:

$$
\Xi(A_1 \oplus u(1) \to B_2) = \{(0,0), (1,0), (2,1), (2,2)\}
$$

Thus the set $\Gamma(A_1 \oplus u(1) \to B_2)$ contains the weight diagrams of $A_1 \oplus u(1)$-representations $\Phi^{\xi_1} = ([1], 1)$, $\Phi^{\xi_2} = ([1], 3)$, $\Phi^{\xi_s} = ([0], 4)$. (The $u(1)$-generator is normalized to have integer eigenvalues.)

$$\Gamma = (\bigcup_{\xi \in \Xi} \Phi^{\xi}) \setminus \{0\} = \{\gamma^{(sign\gamma)}\} = \{(1,0)^{(+)}, (0,1)^{(+)}, (2,1)^{(-)}, (1,2)^{(-)}, (2,2)^{(+)}\}.$$

To simplify the following steps it is convenient to perform further splitting of the diagram of anomalous weights for subrepresentations $\widetilde{L^{\mu}}$. This splitting is not unique, one can choose an arbitrary vector $\varepsilon \in C$ and the projections $a(\kappa)$ are obtained as the scalar products

$$\langle (\kappa - \lambda), -\varepsilon \rangle = a(\kappa)$$

for every κ from to the weight lattice P_g. The weight κ is said to belong to the level $a(\kappa)$. The ordering for the components in Ψ^{λ} thus induced guarantees an inambiguous level by level application of the recursion formula (18). If $\Delta \setminus \widetilde{\Delta}$ contains no positive roots orthogonal to the boundary of $\overline{C}$, the auxiliary vector ε may be placed in the closure of C as well.

Consider the irreducible representation L^{λ} of B_2 with the highest weight $\lambda = (5/2, 1/2)$. The corresponding anomalous weight diagram Ψ^{λ} contains 8 vectors:

$$\Psi^{(5/2,1/2)} = \{\psi^{\epsilon(w)}\} =$$
$$\{(5/2, 1/2)^{(+)}, (-1/2, 7/2)^{(-)}, (5/2, -3/2)^{(-)}, (-5/2, 7/2)^{(+)}, (-1/2, -9/2)^{(+)},$$
$$(-11/2, 1/2)^{(-)}, (-5/2, -9/2)^{(-)}, (-11/2, -3/2)^{(+)}\}.$$

This is the case when splitting can be simplified. One can choose $\varepsilon = (1, 1) \in \overline{C}$ so that $\langle \kappa, \varepsilon \rangle$ becomes proportional to the eigenvalues of $u(1)$-generator in L^{λ} representation. Applying the factorized formula to obtain the decomposition of L^{λ} we are interested in κ's within the Weyl chamber $\overline{C}$. Thus for our example only the weights with nonnegative projection on $\alpha_1 = (1, -1)$ may have positive multiplisities n_{κ}. On the zeroth level the result is trivial

$$\widetilde{n}_{(5/2,1/2)} = n_{(5/2,1/2)} = 1,$$

$$\widetilde{n}_{(-1/2,7/2)} = -1.$$

On the next level (called the first) one finds two points in $\overline{C}$, $(5/2, -1/2)$ and $(3/2, 1/2)$, where the formula (18) gives nonzero values for $\widetilde{n}_{\kappa}$.

$$n_{(5/2,-1/2)} = 1, \quad n_{(3/2,1/2)} = 1.$$

Similarly on the following two levels one finds

$$n_{(1/2,1/2)} = 1, \quad n_{(3/2,-1/2)} = 2$$

and

$$n_{(1/2,-1/2)} = 2, \quad n_{(3/2,-3/2)} = 1.$$

Due to the (reflection) symmetry of the weight diagram these four levels give the sufficient information to write down the final result:

$$[2,1]_{\downarrow A_1 \oplus u(1)} =$$
$$([2],3) \oplus ([1],2) \oplus ([3],2) \oplus 2([2],1) \oplus ([0],1) \oplus 2([1],0) \oplus ([3],0)$$
$$\oplus 2([2],-1) \oplus ([0],-1) \oplus ([3],-2) \oplus ([1],-2) \oplus ([2],3).$$

Here the numbers in the square brackets are the Dynkin indices and the last term in the parenthesis is the eigenvalue of $u(1)$-generator. Note that performing this reduction we need not take into account the anomalous weights outside of the dominant chamber $\overline{C}$. Such additional decomposition of recurrence property takes place only when the vectors of the form $\{\gamma + \xi \mid \xi \in \overline{\widetilde{C}}, \gamma \in \Gamma\}$ do not reach the domain of anomalous weights of $\widetilde{g}$ in $P_g \setminus (P_g \cap \overline{\widetilde{C}})$.

In [7] an attempt was made to achieve the additional decomposition of recurrence property in the situation when the previous condition fails. The injection $A_3 \oplus u(1) \to D_4$ was studied and the recurrence relation connecting only relative multiplicities was obtained. It is slightly different from that described by the formula (18). But one faces great difficulties trying to obtain such algorithm for other pairs of algebras.

In the appendix we bring the more complicated example considering on the injection $A_3 \oplus u(1) \to B_4$. This demonstrates the efficiency of the decomposition algorithm based on the factorized recurrence formula (18). The whole computation is relatively simple and can be easily computerized. The nonmaximal regular injections and special injections can be treated similarly. The detailed study of recurrence relations in these cases will be presented in the forthcoming publication.

Acknowledgments

Appendix.
The Injection $A_3 \oplus u(1) \to B_4$.

The positive root systems for this maximal regular injection in the standard e-basis can be written as follows:

$$
\begin{aligned}
\Delta(B_4) &= \{e_i, e_j - e_k, e_j + e_k\} \\
\widetilde{\Delta}(A_3) &= \{e_j - e_k\} \\
\Delta \setminus \widetilde{\Delta} &= \{e_i, e_j + e_k\} \\
& \quad i, j, k = 1, \ldots, 4; \;\; j < k
\end{aligned}
$$

According to Lemma 2 the set Ξ contains 16 irreducible representations of $A_3 \oplus u(1)$ enumerated by their highest weights ξ_i $(i = 0, \ldots, 15)$:

$$
\Xi(A_3 \oplus u(1) \to B_4) =
$$

$$
\{(0,0,0,0); (1,0,0,0); (2,1,0,0); (2,2,0,0); (3,1,1,0);
$$

$$
(3,2,1,0); (3,3,2,0); (3,3,3,0); (4,1,1,1); (4,2,1,1);
$$

$$
(4,3,2,1); (4,3,3,1); (4,4,2,2); (4,4,3,2); (4,4,4,3);
$$

$$
(4,4,4,4)\}.
$$

In the "fan" $\Gamma(A_3 \oplus u(1) \to B_4)$ the weights γ for each Φ^{ξ_s} bear the same sign, thus the signs can be attributed to the representations $\widetilde{L^{\xi_s}}$:

$$
\{\widetilde{L^{\xi_s}}; \mathrm{sign}(\gamma)\}_{s=1,\ldots,15} =
$$

$$
\{((([1,0,0],1);(+)), ((([1,1,0],3);(-)), ((([0,2,0],4);(+)), ((([2,0,1],5);(+)),
$$

$$
((([1,1,1],6);(-)), ((([3,0,0],7);(-)), ((([0,1,2],8);(+)), ((([2,1,0],8);(+)),
$$

$$
((([0,0,3],9);(-)), ((([1,1,1],10);(-)), ((([1,0,2],11);(+)), ((([0,2,0],12);(+)),
$$

$$
((([0,1,1],13);(-)), ((([0,0,1],15);(+)), ((([0,0,0],16);(-))\}.
$$

The nontrivial multiplicities of $\gamma \in \Gamma$ must also be taken into account. To obtain the splitting one can chose the vector $\varepsilon = (1,1,1,1)$. We write down explicitly only those weights γ that describe the first and the second levels of decomposition:

$$
\Gamma(A_3 \oplus u(1) \to B_4) = \bigcup_{\xi_s, s=1,\ldots,15} \Phi^{\xi_s; \, \mathrm{sign}(\gamma)} =
$$

$$
\{(1,0,0,0)^{(+)}, (0,1,0,0)^{(+)}, (0,0,1,0)^{(+)}, (0,0,0,1)^{(+)}, (2,1,0,0)^{(-)},
$$

$$
(2,0,1,0)^{(-)}, (2,0,0,1)^{(-)}, (1,0,0,2)^{(-)}, (0,1,0,2)^{(-)}, (0,0,1,2)^{(-)},
$$

$$
(1,2,0,0)^{(-)}, (0,2,1,0)^{(-)}, (0,2,0,1)^{(-)}, (1,0,2,0)^{(-)}, (0,1,2,0)^{(-)},
$$

$$
(0,0,2,1)^{(-)}, 2(1,1,1,0)^{(-)}, 2(1,1,0,1)^{(-)}, 2(1,0,1,1)^{(-)}, 2(0,1,1,1)^{(-)}, \ldots
$$

$$
\ldots, (4,4,4,4)^{(-)}\}.
$$

Consider for example the irrep $L^\lambda(B_4)$ with $\lambda = (2,2,1,0)$; dim $L^\lambda = (\underline{1650})$. It has 11 levels. Due to the reflection symmetry $((k_1, k_2, k_3, k_4) \leftrightarrow (-k_1, -k_2, -k_3, -k_4))$ of the weight diagram it is sufficient to study only 6 of them.

Contrary to the previous case the recurrence procedure cannot be performed separately for the relative multiplicities n_κ, that is for $\widetilde{n_\kappa}$ with $\kappa \in \overline{C}$. Nevertheless the calculations involving the anomalous weights (in other Weyl chambers) can be simplified considerably thanks to the following two considerations:

- In the level by level recursive procedure after the evaluation of $\tilde{n}$ in $\overline{\overline{C}}$ the anomalous relative weights in other points can be obtained using the Weyl group V.

- Using the exterior contour of Γ one can easily fix the domain of the weight space that can contribute to the relative multiplicities n_κ and pay no attention to the weights out of this domain.

For the injection $A_3 \oplus u(1) \rightarrow B_4$ only the anomalous weights with nonnegative first coordinate must be taken into account to obtain $\tilde{n} \in \overline{\overline{C}}$. Thus for every obtained highest weight μ one must find only twelve anomalous points of Ψ^μ to be able to proceed the recursion on the next level. On the zeroth level besides the highest weight

$$n_{(2,2,1,0)} = \tilde{n}_{(2,2,1,0)} = 1$$

one must also calculate (using the Weyl group V) the anomalous relative multiplicities:

$$\tilde{n}_{(2,2,-1,2)} = -1, \ \ \tilde{n}_{(2,0,3,0)} = -1, \ \ \tilde{n}_{(2,0,-1,4)} = 1, \ \ \tilde{n}_{(2,-2,3,2)} = 1,$$

$$\tilde{n}_{(2,-2,1,4)} = -1, \ \ \tilde{n}_{(1,3,1,0)} = -1, \ \ \tilde{n}_{(1,3,-1,2)} = 1, \ \ \tilde{n}_{(1,0,4,0)} = 1,$$

$$\tilde{n}_{(1,0,-1,5)} = -1, \ \ \tilde{n}_{(1,-2,4,2)} = -1, \ \ \tilde{n}_{(1,-2,1,5)} = 1.$$

After this the formula (18) can be directly applied to obtain the anomalous relative multiplicities of the first level and among them –

$$n_{(2,2,0,0)} = 1, \ \ n_{(2,1,1,0)} = 1,$$

The 24 anomalous points of these two representations fix the decomposition on the second level:

$$n_{(2,1,0,0)} = 2, \ \ n_{(1,1,1,0)} = 1, \ \ n_{(2,1,1,-1)} = 1, \ \ n_{(2,2,0,-1)} = 1,$$

and so on. For example, on the third level for the weight $(1,1,0,0)$ the formula (18) leads to the following relation:

$$\tilde{n}_{(1,1,0,0)} = n_{(1,1,0,0)} = n_{(1,1,1,0)} + n_{(2,1,0,0)} - 2n_{(2,2,1,0)} - \tilde{n}_{(1,3,1,0)} = 2.$$

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The final result is

$$[0,1,1,0]_{\downarrow A_3 \oplus u(1)} =$$
$$([0,1,1],5) \oplus ([0,2,0],4) \oplus ([1,0,1],4) \oplus 2([1,1,0],3)$$
$$\oplus([0,0,1],3) \oplus ([1,0,2],3) \oplus ([0,2,1],3) \oplus ([2,0,0],2)$$
$$\oplus2([0,1,0],2) \oplus ([0,0,2],2) \oplus 2([1,1,1],2) \oplus 2([1,0,0],1)$$
$$\oplus2([2,0,1],1) \oplus 3([0,1,1],1) \oplus ([1,1,2],1) \oplus ([1,2,0],1)$$
$$\oplus([0,0,0],0) \oplus 3([1,0,1],0) \oplus 2([0,2,0],0) \oplus ([2,0,2],0)$$
$$\oplus([2,1,0],0) \oplus ([0,1,2],0) \oplus 2([1,0,2],-1) \oplus 2([0,0,1],-1)$$
$$\oplus2([1,1,1],-2) \oplus ([2,0,0],-2) \oplus 2([0,1,0],-2) \oplus ([0,0,2],-2)$$
$$\oplus([1,2,0],-3) \oplus ([2,0,1],-3) \oplus ([1,0,0],-3) \oplus 2([0,1,1],-3)$$
$$\oplus([1,0,1],-4) \oplus ([0,2,0],-4) \oplus ([1,1,0],-5).$$

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